Sample Midterm Exam

Part I: Selected Response (20 marks)
Complete each of the following multiple choice items and place the appropriate answer on the answer sheet provided.

C 1. If $f(x) = x + 2$ and $g(x) = x^2 + 6x + 8$, which is the graph of $\left(\frac{g}{f}\right)(x)$?

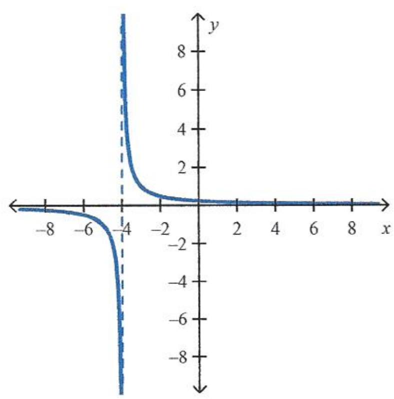
$\frac{g(x)}{f(x)} = \frac{x^2+6x+8}{x+2}$

$y = \frac{(x+4)\cancel{(x+2)}}{\cancel{x+2}}$

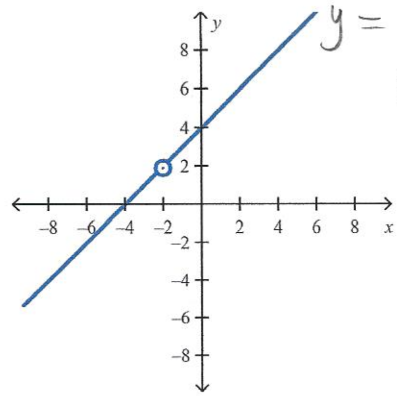
$y = (x+4)$ (linear)

POD $x = -2$

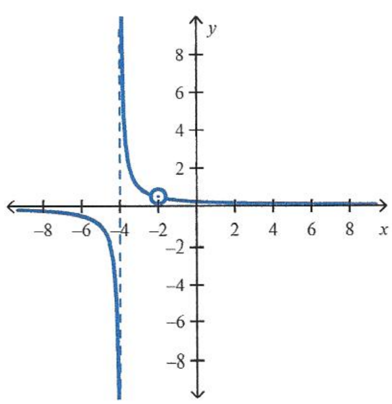
A)



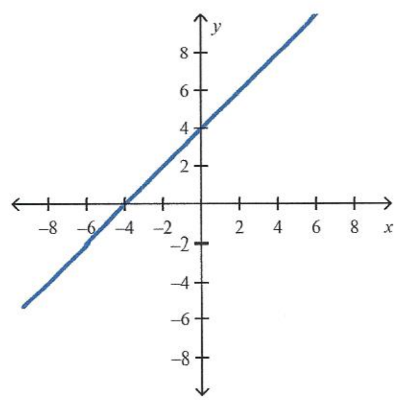
C)



B)



D)



A

2. Given $h(x) = 4x - 4$ and $k(x) = -2x^2 - 4x - 5$, which is the range of $(h-k)(x)$?

A) $\{y | y \geq -7, y \in \mathbb{R}\}$

B) $\{x | x \geq -2, x \in \mathbb{R}\}$

C) $\{x | x \leq -2, y \in \mathbb{R}\}$

D) $\{y | y \leq -7, y \in \mathbb{R}\}$

$h(x) - k(x) = (4x-4) - (-2x^2 - 4x - 5)$

$= 2x^2 + 8x + 1$ parabola opens $\boxed{\text{up}}$

Vertex

$x = \frac{-b}{2a} = \frac{-8}{4} = -2$ $\boxed{V(-2, -7)}$

$y = 2(-2)^2 + 8(-2) + 1 = -7$

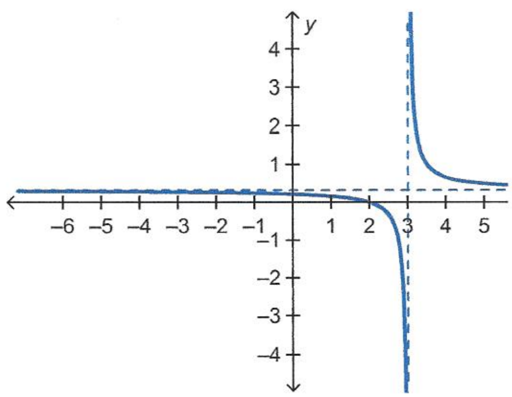
B

3. Which is the graph of $f(x) = \dfrac{x^2 - 4}{3x^2 + 15x + 18}$ $= \dfrac{(x+2)(x-2)}{3(x^2+5x+6)} = \dfrac{(x+2)(x-2)}{3(x+3)(x+2)}$

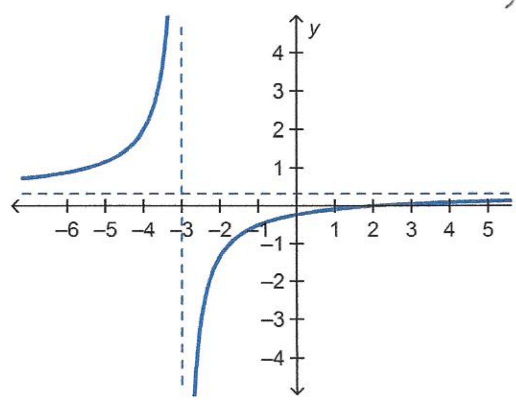
$\Rightarrow$ POD $x = -2$
VA $x = -3$

$(-2, -1.33)$

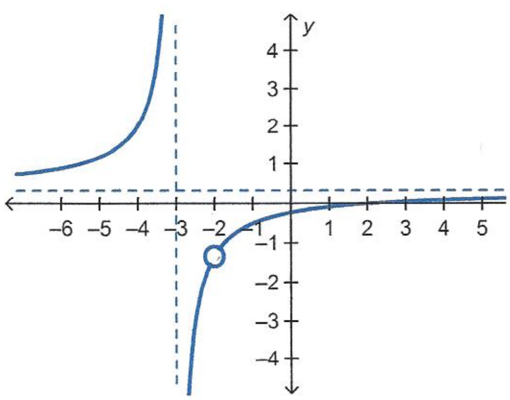
A)



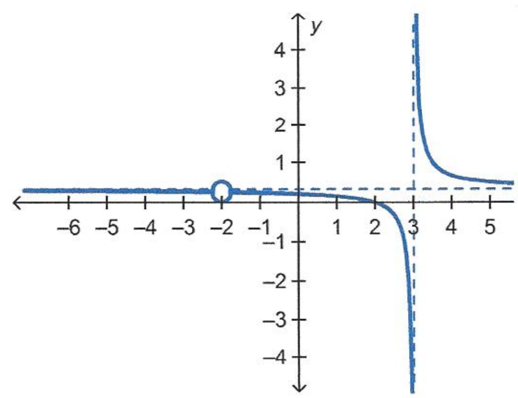
C)



B)



D)



C 4. What is $\lim_{x \to 2} f(x)$?

x	$f(x)$
0	–14
1	–21
1.5	–26.75
1.99	–33.8403
1.999	–33.984003
1.9999	–33.99840003
2.0001	–34.00160003
2.001	–34.016003
2.01	–34.1603
2.5	–42.75
3	–53
4	–78

check

$\lim_{x \to 2^+} f(x) = -34$

$\lim_{x \to 2^-} f(x) = -34$

A) 2

B) ∞

C) –34

D) Does not exist

C 5. Find $\lim_{x \to 2} f(x)$ given $f(x) = \begin{cases} -2x^2 + 4x - 5, & x \le 2 \\ -1x - 3, & x > 2 \end{cases}$

$\lim_{x \to 2^+} (-1x-3) = -5$

$\lim_{x \to 2^-} (-2x^2+4x-5) = -8+8-5 = -5$

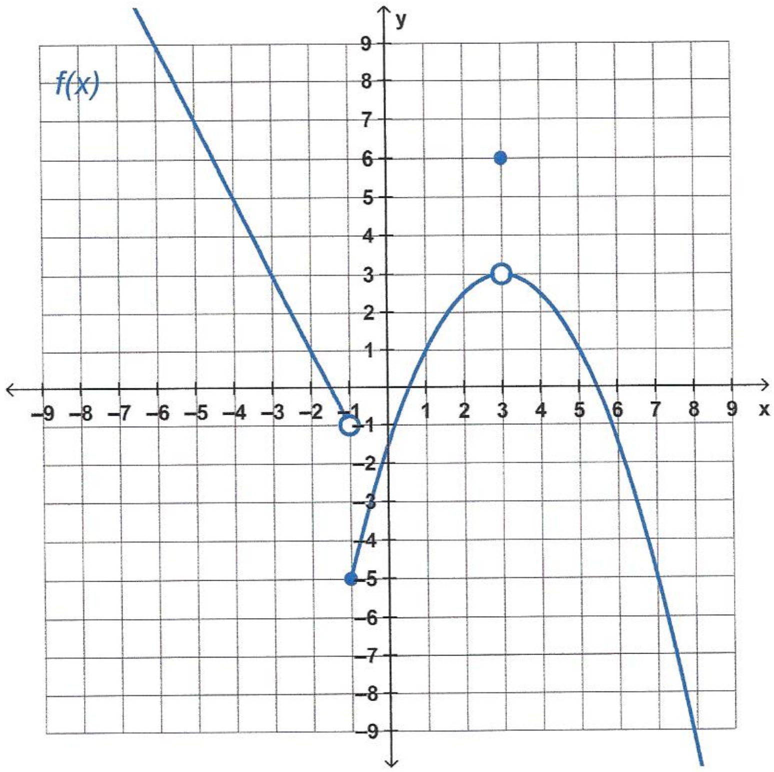
A) 2

B) 5

C) –5

D) Does not exist

C 6. Evaluate $\lim_{x \to -1^+} f(x)$



A) 6

B) −1

C) −5

D) Does not exist

B 7. Evaluate the limit: $\lim_{h \to -1} (h^4 - h^3 - 4h + 6)$ Substitution

A) -12
B) 12
C) -1
D) 6

$(-1)^4 - (-1)^3 - 4(-1) + 6$

$1 - (-1) + 4 + 6$

12

A

8. Evaluate $\lim\limits_{x\to 3}\dfrac{1}{\sqrt{7x^2-6x-9}}$

substitution

A) $\dfrac{1}{6}$

B) $\dfrac{1}{18}$

C) 18

D) Does not exist

$= \dfrac{1}{\sqrt{7(3)^2-6(3)-9}}$

$= \dfrac{1}{\sqrt{63-18-9}} = \dfrac{1}{\sqrt{36}} = \dfrac{1}{6}$

A

9. Which must be true if $f(x)$ is continuous at $x=4$?

A) $\lim\limits_{x\to 4} f(x) = f(4)$

B) $\lim\limits_{x\to \infty} f(x) = f(4)$

C) $\lim\limits_{x\to 4} f(x) = 4$

D) $\lim\limits_{x\to 4} f(x) = \infty$

A

10. Which is continuous at $x=-\frac{1}{2}$?

$f(-\tfrac{1}{2}) = \lim\limits_{x\to -\frac{1}{2}} f(x)$

A) $s(x)=\begin{cases}10x-5, & x<-\frac{1}{2}\\ \frac{5}{x} & x\geq -\frac{1}{2}\end{cases}$

C) $p(x)=\begin{cases}\frac{1}{x+5}, & x<-\frac{1}{2}\\ 1-10x & x\geq -\frac{1}{2}\end{cases}$

B) $m(x)=\begin{cases}10x-5, & x<-\frac{1}{2}\\ \frac{10}{x} & x\geq -\frac{1}{2}\end{cases}$

D) $q(x)=\begin{cases}x^2-5, & x<-\frac{1}{2}\\ 10 & x\geq -\frac{1}{2}\end{cases}$

$\lim\limits_{x\to -\frac{1}{2}^+}\left(\frac{5}{x}\right) = -10$

$f(-\tfrac{1}{2}) = \dfrac{5}{-\frac{1}{2}} = -10$

$\lim\limits_{x\to -\frac{1}{2}^-}(10x-5) = -10$

D 11. Which is (are) the horizontal asymptote(s) for $f(x) = \dfrac{10x^2}{\sqrt{49x^4 - 12}}$?

A) $y = -\dfrac{10}{7}$ and $y = \dfrac{10}{7}$

B) $y = -\dfrac{10}{\sqrt{37}}$

C) $y = \dfrac{10}{\sqrt{37}}$

D) $y = \dfrac{10}{7}$

$$\lim_{x\to\infty} \frac{10x^2}{\sqrt{x^4(49 - 12/x^4)}} = \lim_{x\to\infty} \frac{10x^2}{x^2\sqrt{(49 - 12/x^4)}}$$

$$\lim_{x\to\infty} \frac{\frac{10x^2}{x^2}}{\frac{x^2}{x^2}\sqrt{49 - 12/x^4}} = \frac{10}{1\sqrt{49-0}} = \frac{10}{7}$$

$\lim_{x\to-\infty} f(x)$ is also equal to $10/7$

D 12. Determine the vertical asymptotes of the function $f(x) = \dfrac{4x^2+1}{3x-4x^2}$

A) $x = 0$

B) $x = \dfrac{3}{4}$

C) $x = 0, -\dfrac{3}{4}$

D) $x = 0, \dfrac{3}{4}$

$f(x) = \dfrac{4x^2+1}{x(3-4x)}$

$\boxed{\text{VA } x=0,\ x=\frac{3}{4}}$

$x=0 \quad 3-4x=0$

$\dfrac{3}{4} = \dfrac{4x}{4}$

$\dfrac{3}{4} = x$

A 13. Given $f(x)=2(5x+1)^{-3}$, what is $f'(x)$?

A) $f'(x) = \dfrac{-30}{(5x+1)^4}$

B) $f'(x) = \dfrac{-6}{(5x+1)^4}$

C) $f'(x) = 30(5x+1)^2$

D) $f'(x) = 6(5x+1)^2$

$f'(x) = -6(5x+1)^{-4}(5)$

$f'(x) = \dfrac{-30}{(5x+1)^4}$

C 14. The position of a particle, s, in metres, with respect to time, t, in seconds can be described by the function; $s(t) = 15 - 4t + 3t^2$.
What is the instantaneous speed of the particle at $t = 10$?

A) –275 m/s
B) –56 m/s
C) 56 m/s
D) 275 m/s

$v(t) = -4 + 6t$

$v(10) = -4 + 6(10) = 56$ M/s

C 15. Which is true if f is a function such that $\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = 0$?

A) The limit of $f(x)$ as x approaches 2 does not exist

B) f is not defined at $x = 2$

C) The derivative of f at $x = 2$ is 0

D) f is continuous at $x = 0$

A 16. Determine the x-coordinates of the point where the tangent line to the curve $y = \frac{3}{2}x^4 - 6x^2 + 5$ are horizontal. $\Rightarrow$ Slope $= 0$

$f'(x) = 0$

A) $x = 0, \sqrt{2}, -\sqrt{2}$

B) $x = 0, \sqrt{2}$

C) $x = 0, 2, -2$

D) $x = 0, 2$

$y' = 6x^3 - 12x$

$0 = 6x^3 - 12x$

$0 = 6x(x^2 - 2)$

$6x = 0 \qquad x^2 - 2 = 0$

$x = 0 \qquad x = \pm\sqrt{2}$

B 17. Given $f(x) = g(x) \cdot h(x)$, $f'(-4) = -7$, $g(-4) = -5$, $h(-4) = -1$ and $h'(-4) = -2$, what is the value of $g'(-4)$?

A) -3

B) 17

C) 18

D) 38

$f(x) = g(x) \cdot h(x)$

$f'(x) = g(x)\,h'(x) + h(x)\,g'(x)$

$f'(-4) = g(-4)\,h'(-4) + h(-4)\,g'(-4)$

$-7 = (-5)(-2) + (-1)\,g'(-4)$

$-7 - 10 = (-1)\,g'(-4)$

$-17 = -g'(-4)$

$g'(-4) = 17$

C 18. What is the slope of the tangent line to $f(x) = \dfrac{1}{x-6}$ at $x = 1$?

$f'(x)$

A) 25

B) -5

C) $-\dfrac{1}{25}$

D) $-\dfrac{1}{5}$

$f(x) = (x-6)^{-1}$

$f'(x) = -1(x-6)^{-2}(1)$

$f'(x) = \dfrac{-1}{(x-6)^2}$ $x=1$

$f'(1) = \dfrac{-1}{25}$

C 19. What is the value of $f'\left(\dfrac{1}{2}\right)$ if $f(x) = \dfrac{x}{2}(3x^2 - 5)$?

A) $-\dfrac{17}{16}$

B) $-\dfrac{1}{4}$

C) $-\dfrac{11}{8}$

D) $-\dfrac{1}{8}$

$f(x) = \dfrac{3x^3}{2} - \dfrac{5x}{2}$

$f'(x) = \dfrac{9x^2}{2} - \dfrac{5}{2}$

$f'(\tfrac{1}{2}) = \dfrac{9(\frac{1}{2})^2}{2} - \dfrac{5}{2}$

$f'(\tfrac{1}{2}) = \dfrac{9(\frac{1}{4})}{2} - \dfrac{5}{2}$

$f'(\tfrac{1}{2}) = \dfrac{9}{8} - \dfrac{5}{2}$

$f'(1/2) = \dfrac{9}{8} - \dfrac{20}{8} = -11/8$

D 20. Determine $f'(x)$ for the function $f(x) = x^2 + (x^2 - 1)^5$

A) $f'(x) = 12x(x^2 - 1)^4$

B) $f'(x) = 7x(x^2 - 1)^4$

C) $f'(x) = 2x + 5(x^2 - 1)^4$

D) $f'(x) = 2x + 10x(x^2 - 1)^4$

$f(x) = x^2 + (x^2-1)^5$

$f'(x) = 2x + 5(x^2-1)^4(2x)$

$f'(x) = 2x + 10x(x^2-1)^4$

Name: ____________________

Part II - Constructed Response: Answer all questions in the space provided. (50 marks)

21. Algebraically analyze using intercepts, asymptotes, points of discontinuity, limits, sign diagram and test points to sketch a labelled graph of $y = \dfrac{3x^3 + x^2 - 10x}{x^2 + x - 2}$.

(10 marks)

let $y=0$ x-intercept(s): $\boxed{x=0,\ x=5/3}$

Point(s) of Discontinuity: $x=-2$ $(-2, -2\tfrac{2}{3})$ $\boxed{(-2, -7.3)}$

let $x=0$ y-intercept: $\boxed{(0, 0)}$

Asymptotes: $\boxed{VA\ x=1}$ $\lim_{x\to 1^-} f(x) = \infty$ $\lim_{x\to 1^+} f(x) = -\infty$

$$y = \frac{x(3x^2+x-10)}{(x+2)(x-1)} = \frac{x(3x-5)\cancel{(x+2)}}{\cancel{(x+2)}(x-1)} = \frac{3x^2-5x}{x-1}$$

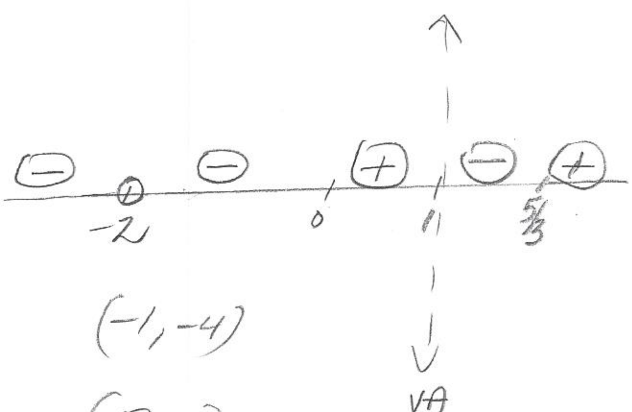
Oblique

$$\begin{array}{r|rrr} 1 & 3 & -5 & 0 \\ & & 3 & -2 \\ \hline & 3 & -2 & -2 \end{array}$$

$\boxed{NOHA}$

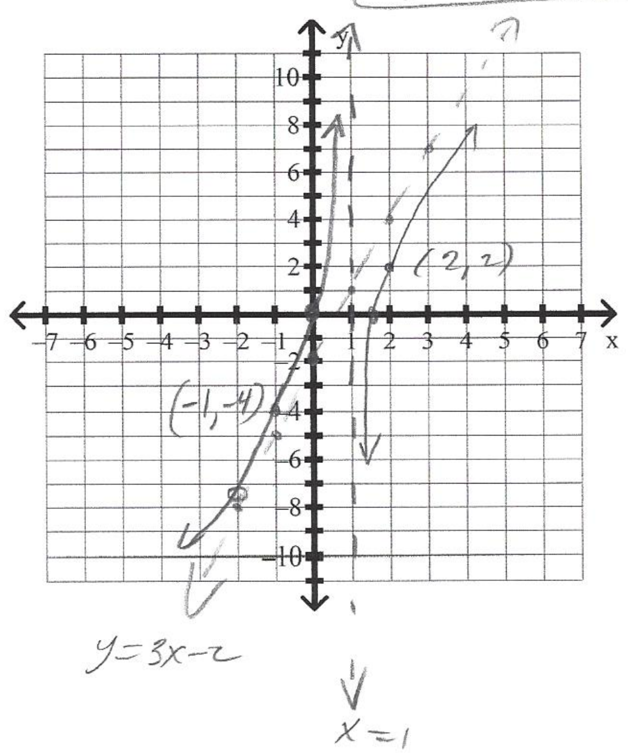
$\boxed{y = 3x-2}$

Sign Diagram:



$(-1, -4)$

$(2, 2)$



22. Algebraically determine each limit.

(A) $\lim_{x \to 3} \frac{x^4 - 27x}{x^2 - 3x} = \lim_{x \to 3} \frac{x(x^3-27)}{x(x-3)} = \lim_{x \to 3} \frac{x(x-3)(x^2-3x+9)}{x(x-3)}$ (3 marks)

$\lim_{x \to 3} (x^2-3x+9) = 27$

(B) $\lim_{x \to 1} \left[\frac{2}{1-x^2} - \frac{1}{1-x}\right] = \lim_{x \to 1} \left[\frac{2}{(1-x)(1+x)} - \frac{1}{1-x}\right]$ (3 marks)

$= \lim_{x \to 1} \left[\frac{2}{(1-x)(1+x)} - \frac{(1+x)}{(1-x)(1+x)}\right] = \lim_{x \to 1} \left[\frac{2-1-x}{(1-x)(1+x)}\right]$

$= \lim_{x \to 1} \frac{\cancel{(1-x)}}{\cancel{(1-x)}(1+x)} = \frac{1}{2}$

(C) $\lim_{x \to 4} \frac{\sqrt{x+5} - 3}{x - 4} \cdot \left[\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}\right]$ (3 marks)

$= \lim_{x \to 4} \frac{(x+5-9)}{(x-4)(\sqrt{x+5}+3)} = \lim_{x \to 4} \frac{\cancel{(x-4)}}{\cancel{(x-4)}(\sqrt{x+5}+3)}$

$= \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

23.

(a) Algebraically determine (using limits) the horizontal asymptotes for $f(x) = \frac{1-2x^3+3x^4}{5-x^2+2x^4}$ (3marks

$$\lim_{x\to\pm\infty} \frac{1-2x^3+3x^4}{5-x^2+2x^4} = \lim_{x\to\pm\infty} \frac{\frac{1}{x^4} - \frac{2x^3}{x^4} + \frac{3x^4}{x^4}}{\frac{5}{x^4} - \frac{x^2}{x^4} + \frac{2x^4}{x^4}} = \frac{3}{2}$$

$$\boxed{y = 3/2}$$

(b) $\lim_{x\to-\infty} \frac{3x+1}{\sqrt{4x^2+1}}$ (4 *marks*)

$$\lim_{x\to-\infty} \frac{3x+1}{\sqrt{x^2(4+1/x^2)}} = \lim_{x\to-\infty} \frac{3x+1}{|x|\sqrt{4+1/x^2}} = \lim_{x\to-\infty} \frac{3x+1}{-x\sqrt{4+1/x^2}}$$

$$\lim_{x\to-\infty} \frac{3x/x + 1/x}{-x/x\sqrt{4+1/x^2}} = \frac{3}{-1\sqrt{4}} = -\frac{3}{2}$$

24. Consider the function $f(x) = \sqrt{4+2x}$.

Use the definition of a derivative to determine $f'(x)$. (5marks)

$$f'(x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h}$$

$$f(x+h) = \sqrt{4+2(x+h)} = \sqrt{4+2x+2h}$$

$$f'(x) = \lim_{h\to0} \frac{\sqrt{4+2x+2h} - \sqrt{4+2x}}{h}$$

$$f'(x) = \lim_{h\to0} \left[\frac{\sqrt{4+2x+2h} - \sqrt{4+2x}}{h}\right]\left[\frac{\sqrt{4+2x+2h} + \sqrt{4+2x}}{\sqrt{4+2x+2h} + \sqrt{4+2x}}\right]$$

$$f'(x) = \lim_{h\to0} \frac{4+2x+2h-(4+2x)}{h(\sqrt{4+2x+2h}+\sqrt{4+2x})} = \lim_{h\to0} \frac{2\cancel{h}}{\cancel{h}(\sqrt{4+2x+2h}+\sqrt{4+2x})}$$

$\to 0$

$$f'(x) = \frac{2}{2\sqrt{4+2x}} = \frac{1}{\sqrt{4+2x}}$$

25.

Consider the function:

$$f(x) = \begin{cases} \dfrac{6x+12}{x^2-4} & x<-1 \\ 3x+1 & x=-1 \\ \dfrac{x^2-5x+2}{x-3} & x>-1 \end{cases}$$

$\frac{6(x+2)}{(x+2)(x-2)}$ Non permissible $x = \pm 2$ only need to check -2

$\frac{x^2-5x+2}{x-3}$ non permissible $x=3$

Use the definition of continuity to determine all points at which f(x) is not continuous.

Classify any discontinuities as removable or non-removable, and explain your choice. (6 *marks*)

check $x=-1$

i) $f(-1) = 3(-1)+1 = -2$

$\lim\limits_{x\to -1^+} \frac{x^2-5x+2}{x-3} = -2$

$\lim\limits_{x\to -1^-} \frac{6x+12}{x^2-4} = -2$

$f(-1) = \lim\limits_{x\to -1} f(x)$ continuous at $x=-1$

check $x=3$

i) $f(3) = -4/0$ undefined

$\lim\limits_{x\to 3} \frac{x^2-5x+2}{x-3}$ DNE $\left(\frac{c}{0}\right)$

discontinuous at $x=3$ (non-removable)

check $x=-2$

i) $f(-2) = \frac{6(-2)+12}{(-2)^2-4} = \frac{0}{0}$ POD

$\lim\limits_{x\to -2} \frac{6x+12}{x^2-4} = \lim\limits_{x\to -2} \frac{6(x+2)}{(x+2)(x-2)} = -3/2$

discontinuous at $x=-2$ Removable

26. (a) Given $y = \dfrac{6x^2 - 4}{(3x+1)^3}$, show that $\dfrac{dy}{dx} = \dfrac{-6\left[3x^2 - 2x - 6\right]}{[3x+1]^4}$ (5 marks)

$y = (6x^2-4)(3x+1)^{-3}$

$\frac{dy}{dx} = (6x^2-4)\ -3(3x+1)^{-4}(3) + (3x+1)^{-3}(12x)$

$\frac{dy}{dx} = -\frac{9(6x^2-4)}{(3x+1)^4} + \frac{12x}{(3x+1)^3}$

$\frac{dy}{dx} = -\frac{9(6x^2-4)}{(3x+1)^4} + \frac{12x(3x+1)}{(3x+1)^4}$

$\frac{dy}{dx} = \frac{-54x^2+36+36x^2+12x}{(3x+1)^4}$

$\frac{dy}{dx} = \frac{-18x^2+12x+36}{(3x+1)^4}$

$\frac{dy}{dx} = \frac{-6(3x^2-2x-6)}{(3x+1)^4}$

(b) *Determine* $f'(x)$ *if* $f(x) = \sqrt{x + \sqrt{x^2+1}}$ (3 *marks*)

$f(x) = \left[x + (x^2+1)^{\frac{1}{2}}\right]^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}\left[x + (x^2+1)^{\frac{1}{2}}\right]^{-\frac{1}{2}} \cdot \left[1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)\right]$

$f'(x) = \frac{1}{2}\left(x + (x^2+1)^{\frac{1}{2}}\right]^{-\frac{1}{2}} \left[1 + x(x^2+1)^{-\frac{1}{2}}\right]$

$f'(x) = \frac{1}{2\sqrt{x+\sqrt{x^2+1}}}\left[1 + \frac{x}{\sqrt{x^2+1}}\right]$

(c) *Determine* $f'(x)$ *if* $f(x) = (2x+1)^4(3x-5)^7$ (4 *marks*)

$f'(x) = (2x+1)^4\ 7(3x-5)^6(3) + (3x-5)^7\ 4(2x+1)^3(2)$

$f'(x) = 21(2x+1)^4(3x-5)^6 + 8(2x+1)^3(3x-5)^7$

$f'(x) = (2x+1)^3(3x-5)^6\left[21(2x+1) + 8(3x-5)\right]$

$f'(x) = (2x+1)^3(3x-5)^6\left[66x - 19\right]$

27. Find $\frac{dy}{dx}$ if $x^3+3xy+y^3=15$.

Use this to determine the equation of the normal line to the curve at the point (1,2).

(5 marks)

$$3x^2+3\left[x\,dy/dx+y\right]+3y^2\frac{dy}{dx}=0$$

$$3x^2+3x\frac{dy}{dx}+3y+3y^2\frac{dy}{dx}=0$$

$$3x\frac{dy}{dx}+3y^2\frac{dy}{dx}=-3x^2-3y$$

$$\frac{dy}{dx}(3x+3y^2)=-3x^2-3y$$

$$\frac{dy}{dx}=\frac{-3x^2-3y}{3x+3y^2} \qquad (1,2)$$

$$\frac{dy}{dx}=\frac{-3(1)-3(2)}{3(1)+3(2)^2}=\frac{-9}{15}=\frac{-3}{5}$$

$$m_{normal}=\frac{5}{3}$$

$$y-y_1=m(x-x_1)$$

$$y-2=5/3\,(x-2)$$

$$y-2=5/3\,x-10/3$$

$$y=5/3\,x-4/3$$