Calculus 3208

Calculus of Trigonometry

a) $y = \frac{\sin x}{x^2} = (\sin x)x^{-2}$

$\frac{dy}{dx} = (\sin x)(-2x^{-3}) + (x^{-2})\cos x$

$\frac{dy}{dx} = \frac{-2\sin x}{x^3} + \frac{\cos x}{x^2} = \frac{-2\sin x + x\cos x}{x^3}$

b) $y = 3\sin x + \cos\left(\frac{x}{2}\right) - 1$

$y' = 3\cos x - \sin\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)$

$y' = 3\cos x - \frac{1}{2}\sin\left(\frac{x}{2}\right)$

c) $y = \sin\left(\sqrt[3]{2x+1}\right)$

$y = \sin\left[(2x+1)^{\frac{1}{3}}\right]$

$y' = \cos\left[(2x+1)^{\frac{1}{3}}\right] \cdot \left[\frac{1}{3}(2x+1)^{-2/3}(2)\right]$

$y' = \frac{2\cos\left(\sqrt[3]{2x+1}\right)}{3\left(\sqrt[3]{2x+1}\right)^2}$

d) $y = \sin^7 \sqrt{x}$

$y = \left[\sin(x^{\frac{1}{2}})\right]^7$

$y' = 7(\sin(x^{\frac{1}{2}}))^6 \cdot \cos(x^{\frac{1}{2}}) \cdot \frac{1}{2}x^{-\frac{1}{2}}$

$y' = \frac{7\sin^6(\sqrt{x})\cos(\sqrt{x})}{2\sqrt{x}}$

e) $y = \cos x \tan x \csc x$

$y = \cancel{\cos x} \frac{\cancel{\sin x}}{\cancel{\cos x}} \frac{1}{\cancel{\sin x}}$

$y = 1$

$\boxed{y' = 0}$

f) $y = \frac{x \sin x}{\cos x}$

$y' = \frac{\cos x[x \cos x + \sin x] - (x \sin x)(-\sin x)}{\cos^2 x}$

$y' = \frac{x \cos^2 x + \cos x \sin x + x \sin^2 x}{\cos^2 x}$

$y' = \frac{x \overset{=1}{(\cos^2 x + \sin^2 x)} + \cos x \sin x}{\cos^2 x}$

$y' = \frac{x + \cos x \sin x}{\cos^2 x}$

g) $y = (x-5)^8 \cot(7x)$

$y' = (x-5)^8 (-\csc^2(7x) \cdot 7) + \cot(7x) [8(x-5)^7 \cdot (1)]$

$y' = (x-5)^8 (-7 \csc^2(7x)) + 8 (x-5)^7 \cot(7x)$

$y' = (x-5)^7 [7(x-5) \csc^2(7x) + 8 \cot(7x)]$

h) $y = \tan^4 \sqrt{1+3x}$

$y = [\tan (1+3x)^{\frac{1}{2}}]^4$

$y' = 4 [\tan (1+3x)^{\frac{1}{2}}]^3 \cdot \sec^2 (1+3x)^{\frac{1}{2}} \cdot [\frac{1}{2} (1+3x)^{-\frac{1}{2}} (3)]$

$y' = \frac{6 \tan^3 (\sqrt{1+3x}) \sec^2 \sqrt{1+3x}}{\sqrt{1+3x}}$

i) $y = 5\cos^3(1+\tan x)$

$y = 5\,[\cos(1+\tan x)]^3$

$y' = 5 \cdot 3[\cos(1+\tan x)]^2 \cdot (-\sin(1+\tan x) \cdot (\sec^2 x)$

$y' = -15\cos^2(1+\tan x)\sin(1+\tan x)\sec^2 x$

j) $y = \dfrac{\cos x}{1+\sin x}$

$y' = \dfrac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2}$

$y' = \dfrac{-\sin x \mp \sin^2 x - \cos^2 x}{(1+\sin x)^2}$

$y' = \dfrac{-1(\sin x + \sin^2 x + \cos^2 x)}{(1+\sin x)^2}$

$y' = \dfrac{-1(\sin x + 1)}{(1+\sin x)^2}$

$y' = \dfrac{-1}{1+\sin x}$

k) $y = \sec^2 x + \sec(x^2)$

$y = (\sec x)^2 + \sec(x^2)$

$y' = 2(\sec x) \cdot \sec x \tan x + \sec(x^2)\tan(x^2) \cdot 2x$

$y' = 2\sec^2 x \tan x + 2x \sec(x^2)\tan(x^2)$

l) $y = \tan(\frac{1}{x^3})$

$y = \tan(x^{-3})$

$y' = \sec^2(x^{-3}) \cdot (-3x^{-4})$

$y' = \dfrac{-3\sec^2(\frac{1}{x^3})}{x^4}$

m) $y = \tan^2(\pi - x^3)$

$y = [\tan(\pi - x^3)]^2$

$y' = 2[\tan(\pi - x^3)]' \cdot \sec^2(\pi - x^3) \cdot (-3x^2)$

$y' = -6x^2 \tan(\pi - x^3)\sec^2(\pi - x^3)$

n) $y = \dfrac{1 + 2\tan(3x)}{1 - x}$

$y' = \dfrac{(1-x)(2\sec^2(3x) \cdot 3) - (1 + 2\tan(3x))(-1)}{(1-x)^2}$

$y' = \dfrac{6(1-x)\sec^2(3x) + (1 + 2\tan(3x))}{(1-x)^2}$

o) $y = \sqrt{1+\sin 2x}$

$y = (1+\sin 2x)^{\frac{1}{2}}$

$y' = \frac{1}{2}(1+\sin 2x)^{-1/2} \cdot \cos 2x \quad (2)$

$$y' = \frac{2\cos 2x}{2\sqrt{1+\sin 2x}}$$

p) $y = \arctan(3x)$

$$y' = \frac{1}{1+(3x)^2} \cdot 3$$

$$y' = \frac{3}{1+9x^2}$$

q) $y = \sin(\cos^{-1}(x))$

$$y' = \cos(\cos^{-1}(x)) \cdot -\frac{1}{\sqrt{1-x^2}}$$

$$y' = \frac{-\cos(\cos^{-1}(x))}{\sqrt{1-x^2}}$$

r) $y = \cos^{-1}\sqrt{2x-1}$

$$y' = \frac{-1}{\sqrt{1-(\sqrt{2x-1})^2}} \cdot \frac{1}{2}(2x-1)^{-1/2} \cdot (2)$$

$$y' = \frac{-1}{\sqrt{1-(2x-1)}\ \sqrt{2x-1}} = \frac{-1}{\sqrt{2-2x}\ \sqrt{2x-1}}$$

$y' =$

#2. $f(x) = \cos(3x)$

$f'(x) = -\sin(3x)\ (3) = -3\sin 3x$

$f''(x) = -3\cos 3x\ (3) = -9\cos 3x$

$f'''(x) = 9\sin(3x)3 = +27\sin 3x$

$f^{(4)}(x) = 27\cos(3x)\ 3 = \boxed{81 \cos 3x}$

#3a) $\lim\limits_{x\to 0} \frac{\sin^2 4x}{x^2} = \lim\limits_{x\to 0} \frac{\sin 4x}{x} \frac{\sin 4x}{x}$

$= \frac{1}{4}\frac{1}{4} \lim\limits_{x\to 0} \frac{\sin 4x}{4x} \cdot \frac{\sin 4x}{4}$

$= \frac{1}{16}(1)(1)$

$\boxed{= \frac{1}{16}}$

b) $\lim\limits_{x\to 0} \frac{\sin 8x}{4x} = \frac{1}{2} \lim\limits_{x\to 0} \frac{\sin 8x}{(2)\ 4x}$

$= \frac{1}{2}(1)$

$\boxed{= \frac{1}{2}}$

c) $\lim\limits_{x\to 0} \frac{\cos x - 1}{\sin x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$

$= \lim\limits_{x\to 0} \frac{\frac{\cos x - 1}{x}}{\frac{\sin x}{x}} = \frac{0}{1} = \boxed{0}$

3d) $\lim_{x \to 0} \frac{\sin^2 5x}{x^2 \cos^2 2x}$

$= \lim_{x \to 0} \frac{\sin(5x)}{x} \frac{\sin(5x)}{x} \cdot \frac{1}{\cos^2(2x)}$

$= \frac{1}{5} \cdot \frac{1}{5} \lim_{x \to 0} \frac{\sin(5x)}{5x} \frac{\sin(5x)}{5x} \cdot \frac{1}{\cos^2(2x)}$

$= \frac{1}{25}$

3e) $\lim_{x \to 0} \frac{1-\cos^2 x}{x^2(2+\cos x)} = \lim_{h \to 0} \frac{\sin^2 x}{x^2(2+\cos x)}$

$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \frac{1}{2+\cos x}$

$= (1)\ (1)\ \frac{1}{2+1}$

$= \frac{1}{3}$

4a) $y^2 \cos^2(x) = 4$

$y^2 [\cos x]^2 = 4$

$y^2 (2\cos x)(-\sin x) + \cos^2 x \, 2y \frac{dy}{dx} = 0$

$-2y^2 \sin x \cos x + 2y \cos^2 x \frac{dy}{dx} = 0$

$2y \cos^2 x \frac{dy}{dx} = 2y^2 \sin x \cos x$

$\frac{dy}{dx} = \frac{y \sin x \cos x}{\cos^2 x}$

$\frac{dy}{dx} = \frac{y \sin x}{\cos x}$

4b) $x + \sin y = xy$

$x \cdot \cdot \cos y \cdot \frac{dy}{dx} + \sin y \cdot (1) = x\frac{dy}{dx} + y(1)$

$x\cos y \frac{dy}{dx} + \sin y = x\frac{dy}{dx} + y$

$x\cos y \frac{dy}{dx} - x\frac{dy}{dx} = -\sin y + y$

$\frac{dy}{dx}(x\cos y - x) = -\sin y + y$

$$\frac{dy}{dx} = \frac{-\sin y + y}{x\cos y - x}$$

4c) $x^2\cos y = y^2 \sin x$

$x^2(-\sin y)\frac{dy}{dx} + \cos y \cdot (2x) = y^2(\cos x) + (\sin x)\ 2y\frac{dy}{dx}$

$-x^2\sin y \frac{dy}{dx} + 2x\cos y = y^2\cos x + 2y\sin x \frac{dy}{dx}$

$-x^2\sin y \frac{dy}{dx} - 2y\sin x \frac{dy}{dx} = y^2\cos x - 2x\cos y$

$\frac{dy}{dx}(-x^2\sin y - 2y\sin x) = y^2\cos x - 2x\cos y$

$$\frac{dy}{dx} = \frac{y^2\cos x - 2x\cos y}{-x^2\sin y - 2y\sin x}$$

5. $\pi(x^2 - y^2) = \cos(\pi y)$ $\qquad (\frac{3}{2}, \frac{3}{2})$

$$\pi\left(2x - 2y \cdot \frac{dy}{dx}\right) = -\sin(\pi y) \cdot \pi \frac{dy}{dx}$$

$$2\pi x - 2y\pi \frac{dy}{dx} = -\pi \sin(\pi y) \cdot \frac{dy}{dx}$$

$$2\pi x = \left[-\pi \sin(\pi y) + 2y\pi\right] \frac{dy}{dx}$$

$$\frac{2\pi x}{-\pi \sin(\pi y) + 2y\pi} = \frac{dy}{dx}$$

$$\frac{2\cancel{\pi} x}{\cancel{\pi}(-\sin(\pi y) + 2y)} = \frac{dy}{dx}$$

$$\frac{2x}{-\sin(\pi y) + 2y} = \frac{dy}{dx}$$

$$\frac{2(3/2)}{-\sin(\pi \cdot 3/2) + 2(3/2)} = \frac{dy}{dx}$$

$$\frac{3}{-\sin(\frac{3\pi}{2}) + 3} = \frac{dy}{dx} \qquad \sin \frac{3\pi}{2} = -1$$

$$\frac{3}{-(-1) + 3} = \frac{dy}{dx}$$

$$\frac{3}{4} = \frac{dy}{dx}$$

6. $y = \sin x \tan x$ $\qquad x = \pi/6$,

$y = \sin\frac{\pi}{6} \tan\frac{\pi}{6}$

$y = \left(\frac{1}{2}\right)\frac{1}{\sqrt{3}} =$

$y = \frac{1}{2}\frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{6}$

$y' = \sin x \cdot \sec^2 x + \tan x(\cos x)$

$y' = \sin x \sec^2 x + \frac{\sin x}{\cancel{\cos x}} \cdot \cancel{\cos x}$

$y' = \sin x \sec^2 x + \sin x$

$y' = \sin x(\sec^2 x + 1)$ $\qquad x = \pi/6$

$y' = (\sin \pi/6)\ (\sec^2(\pi/6) + 1)$

$y' = \left(\frac{1}{2}\right)\ \left(\frac{4}{3} + 1\right)$

$y' = \left(\frac{1}{2}\right)\left(\frac{7}{3}\right) = \frac{7}{6}$

$\sec^2 \frac{\pi}{6}$

$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\sec \frac{\pi}{6} = \frac{2}{\sqrt{3}}$

$\sec^2 \frac{\pi}{6} = \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}$

tangent line: $y - y_1 = m(x - x_1)$

$y - \frac{\sqrt{3}}{6} = \frac{7}{6}(x - \pi/6)$

normal line: $y - \frac{\sqrt{3}}{6} = \frac{6}{7}(x - \pi/6)$

7. $y = \cos 3x \qquad (\pi/6, 0)$

$y' = -\sin 3x\,(3)$

$y' = -3\sin 3x$

$y' = -3\sin 3(\pi/6)$

$y' = -3\sin(\pi/2)$

$y' = -3(1) = -3$

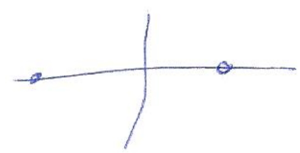
$y - 0 = -3(x - \pi/6)$

tangent line horizontal where $y' = 0$

$-3\sin 3x = 0$

$\sin 3x = 0 \qquad$ let $w = 3x$

$\sin w = 0$



$w = 0 + \pi k \quad k \in I$

$3x = 0 + \pi k \quad k \in I$

$x = 0 + \frac{\pi}{3}k \quad k \in I$

$k = 1 \qquad x = \frac{\pi}{3} \qquad\qquad k = 6 \qquad x = 2\pi$

$k = 0 \qquad x = 0$

$k = -1 \qquad x = \cancel{-\pi/3}$

$k = 2 \qquad x = \frac{2\pi}{3}$

$k = 3 \qquad x = \pi$

$k = 4 \qquad x = \frac{4\pi}{3}$

$k = 5 \qquad x = 5\pi/3$