Applications of Derivatives

Curve Sketching (using derivatives):

A) Polynomial Functions

B) Rational Functions

Lesson 5.2 Use Derivatives to Sketch the Graph of a Polynomial Function.

Idea:

1) Identify intervals of increase and decrease
2) Critical numbers
3) Relative and absolute extrema

} first derivative

4) Identify hypercritical numbers
5) Intervals of concavity
6) Points of inflection

} second derivative

7) x and y-intercepts

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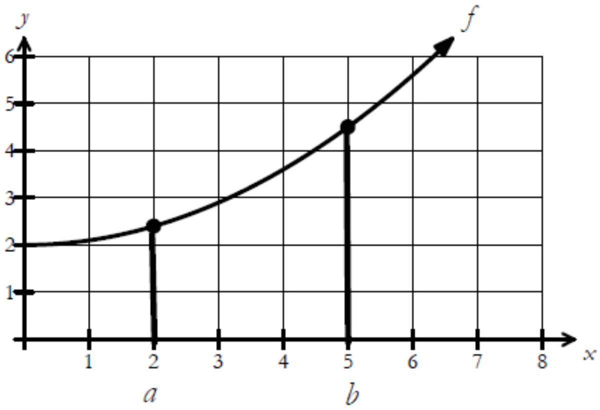
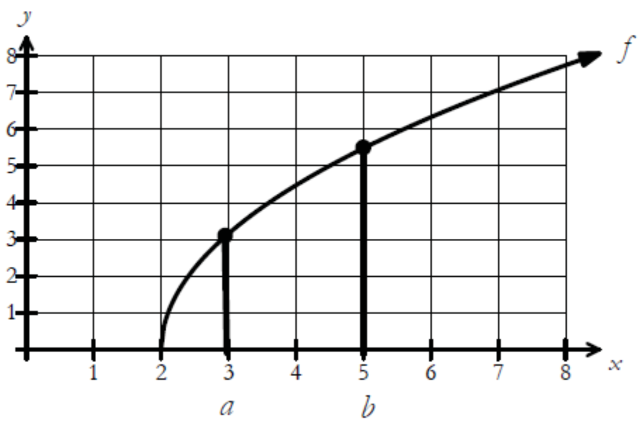
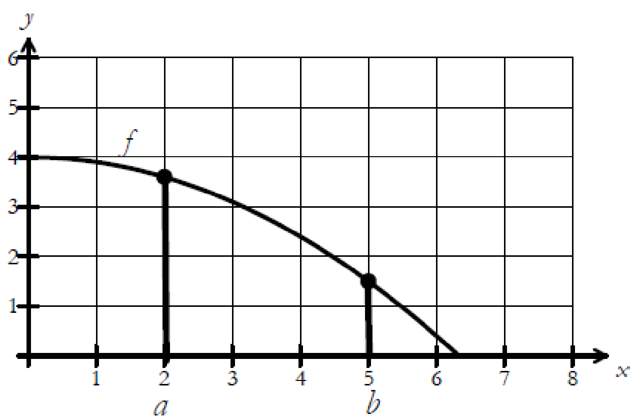
Increasing Functions

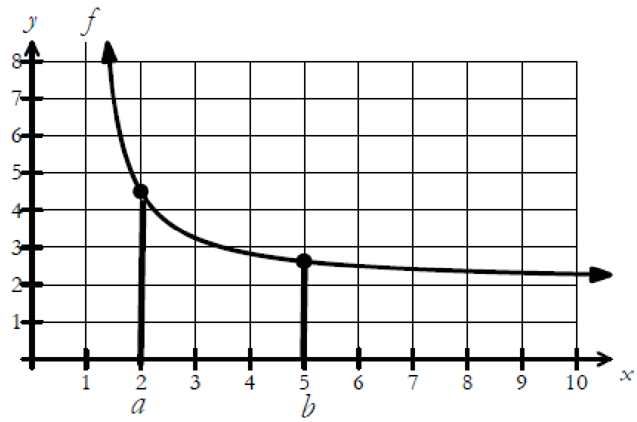




$$f(b) > f(a) \text{ where } b > a$$

Decreasing Functions





$$f(b) < f(a) \text{ where } b > a$$

The Concepts of Increasing/Decreasing Behaviour

On any given interval:

$f'(x) > 0$ → Tangents have positive slopes → $f(x)$ Increases

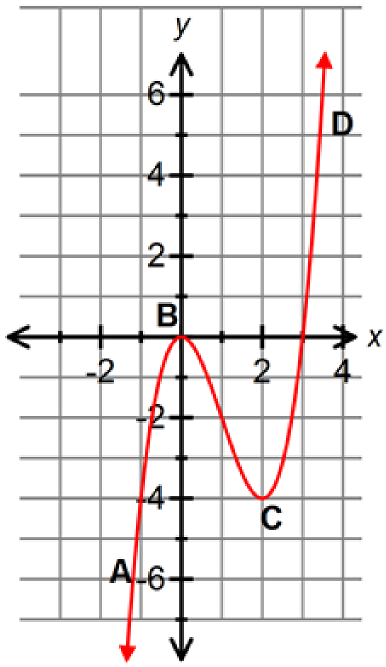
$f'(x) < 0$ → Tangents have negative slopes → $f(x)$ Decreases

$f'(x) = 0$ → Tangents are horizontal → $f(x)$ Max / Min /

A → B

$f(x)$ is ________

$f'(x)$ is ________



B → C
C → D

$f(x)$ is ________

$f'(x)$ is ________

$f(x)$ is ________

$f'(x)$ is ________

Point B: ________________

Point C: ________________

Critical Number (c)

↳ A number c in the domain of $f(x)$ where $f'(c) = 0$ or $f'(c)$ is undefined.

↳ Critical numbers can help determine the local extrema.

First Derivative Test

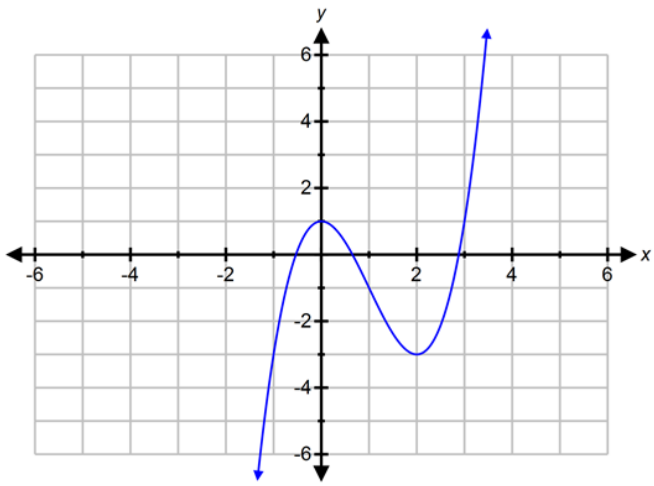
Let c be a critical number:

1) If $f'(x)$ changes from + to - at c, then $f(x)$ has a local maximum at c

2) If $f'(x)$ changes from - to + at c, then $f(x)$ has a local minimum at c

3) If $f'(x)$ does not change sign at c, then $f(x)$ has no local extrema at c

Idea: $y = x^3 - 3x^2 + 1$



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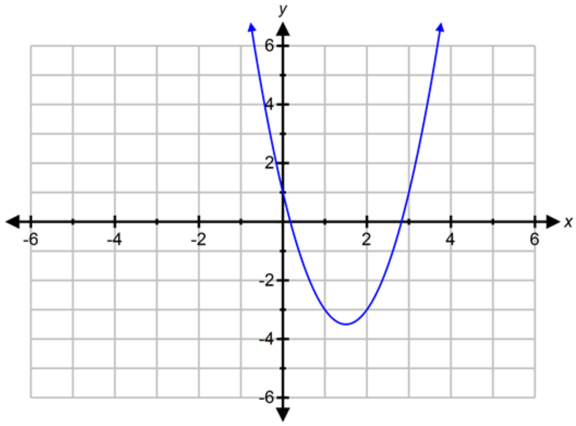
Lesson 5.2 Graph of a Polynomial

Example 1

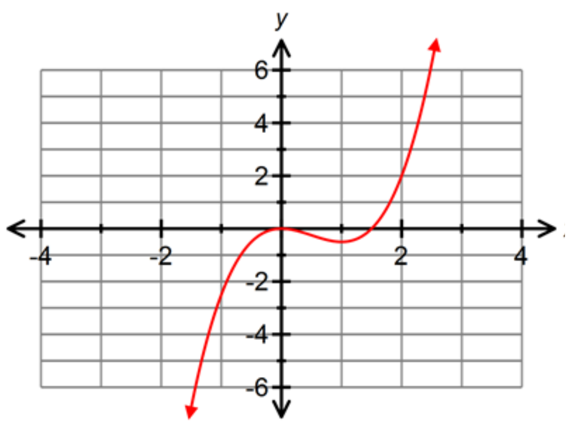
Determine the vertex of $f(x) = 2x^2 - 6x + 1$



Example 2

Algebraically determine where the function $f(x) = x^3 - \frac{3}{2}x^2$ is increasing and decreasing.

Identify the local maximum or minimum.



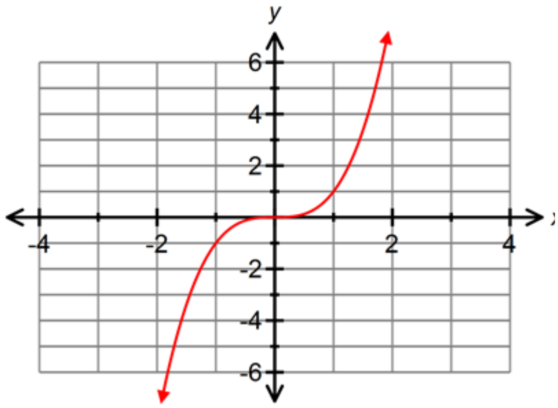
Do all critical numbers give rise to a local maximum or local minimum?

Example 3

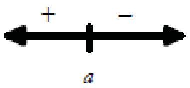
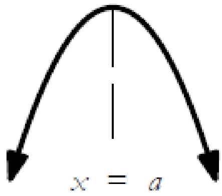
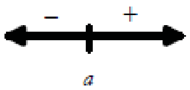
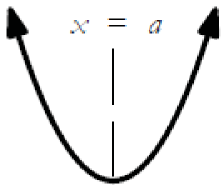
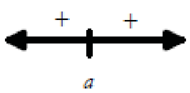
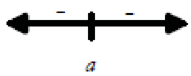
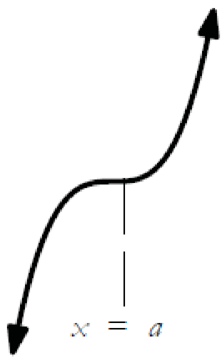
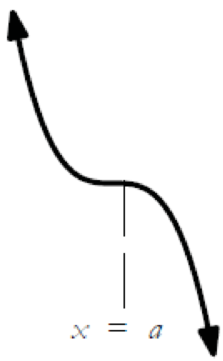
$$f(x) = x^3$$

What is the critical number?

Use a sign diagram to check where the function is increasing and decreasing?



Summary:

sign diagram of $f\,'(x)$ near $x = a$	description	shape of curve near $x = a$
	If $f\,'(x)$ changes from positive to negative at a and $f(a)$ exists, then $f(a)$ is a local maximum.	
	If $f\,'(x)$ changes from negative to positive at a and $f(a)$ exists, then $f(a)$ is a local minimum.	
 	If $f\,'(x)$ does not change sign, then there is neither a local maximum nor local minimum at a.	 

Relative(local) and Absolute Extrema

A: absolute maximum

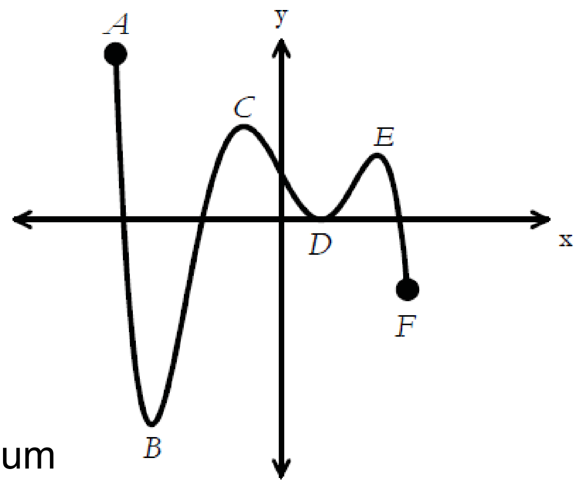
B: both a local and absolute minimum

C: local maximum

D: local minimum

E: local maximum

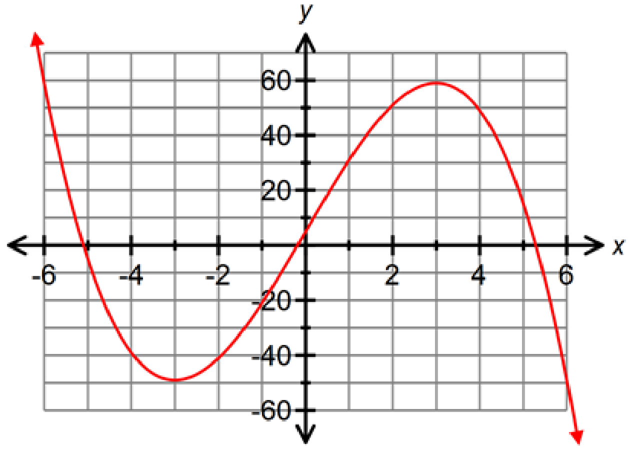
F: neither a local nor an absolute minimum



Critical numbers can be used to determine the absolute maximum and minimum.

Example 4

Determine the absolute maximum and minimum values of $f(x) = -x^3 + 27x + 5$ on the interval [0,4].



Idea:

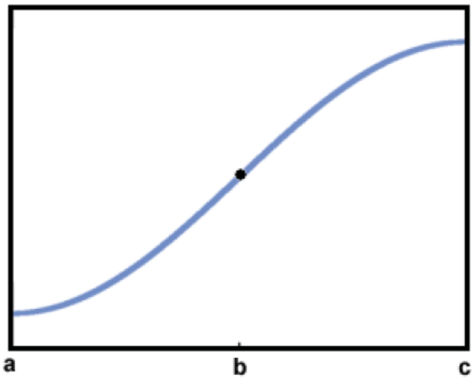
1) Determine the critical numbers in the interval and compute the functional values at these points.

2) Determine the functional values at the endpoints $f(0)$ and $f(4)$

3) The largest value is the absolute maximum; the smallest value is the absolute minimum.

What does f'' say about f ?

Concavity



The function $f(x)$ is increasing on [a,c]. What about the behaviour prior to point b and how is it different after point b?

Where are the tangent lines in relation to the curve?

How are the slopes of the tangent lines changing?

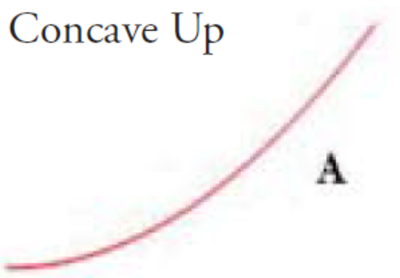
Concave up: the curve lies above its tangent lines

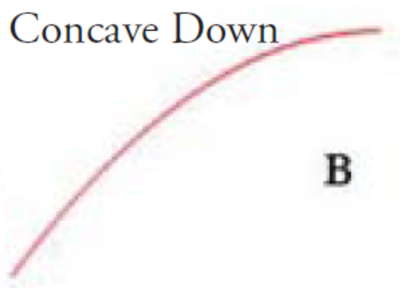
Concave down: the curve lies below its tangent lines

A section of the graph of $f(x)$ is considered to be <u>concave up</u> if its slope ($f'(x)$) increases as x increases $\longrightarrow f''(x) > 0$

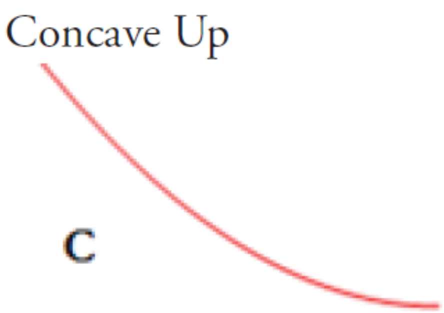
A section of the graph of $f(x)$ is considered to be <u>concave down</u> if its slope ($f'(x)$) decreases as x increases $\longrightarrow f''(x) < 0$

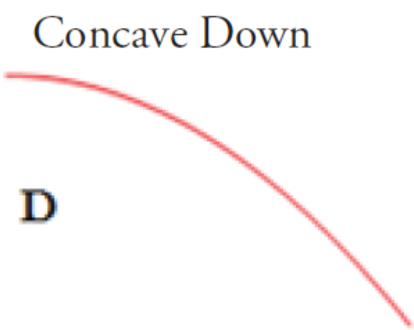
Increasing Function





Decreasing Function





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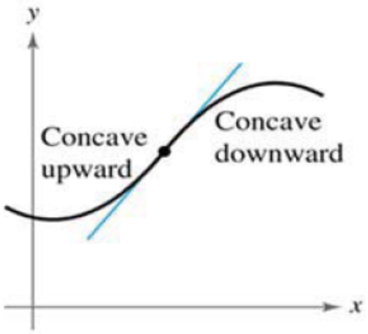
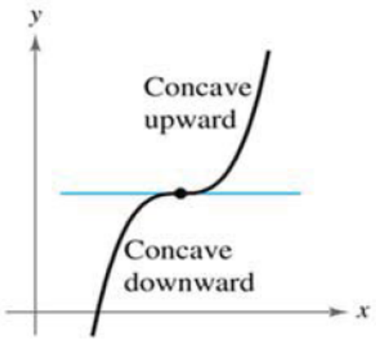
Hypercritical numbers

↳The values in the domain of $f(x)$ where $f''(x)=0$ or $f''(x)$ is undefined.

Point of Inflection

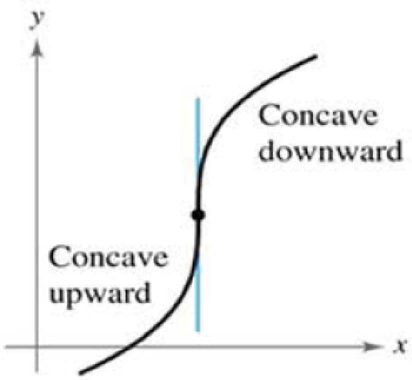
↳ Concavity of a function changes at a point of inflection.





$f''(x)>0$
concave up

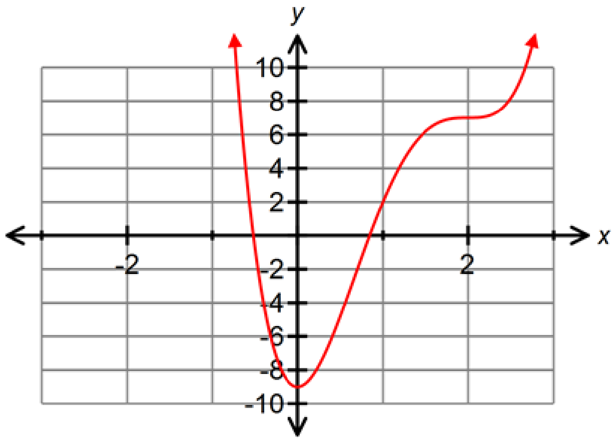
$f''(x)<0$
concave down



If $f''(x)$ changes sign, the hypercritical value is a point of inflection. It marks the transition between concave up and concave down.

Example 5

Determine where the graph of $f(x) = 3x^4 - 16x^3 + 24x^2 - 9$
is concave up and concave down and determine any inflection points.



Example 6

$f(x) = x^3 + 3x^2 - 9x - 11$

State the following:

1) Algebraically determine the x and y-intercepts
2) Determine y' and state the critical numbers
3) State the intervals of increase and decrease
4) Determine the local extrema
5) Determine y'' and state the hypercritical numbers
6) State the intervals of concavity.
7) Determine the points of inflection
8) Sketch the graph.

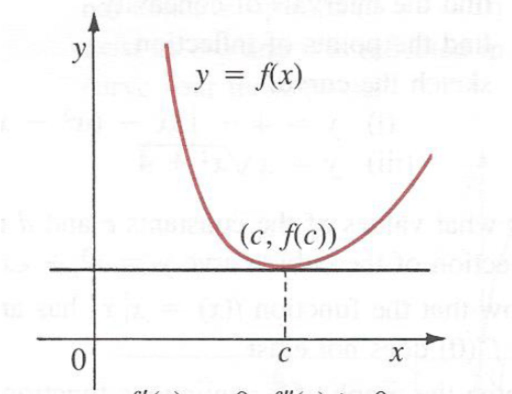
Example 7

For the equation, $y = x(x-4)^3$ determine where the graph is increasing, decreasing, any relative maximum and minimum points, where the graph is concave upward and concave downward, and any inflection points. Sketch a labelled graph including the intercepts.

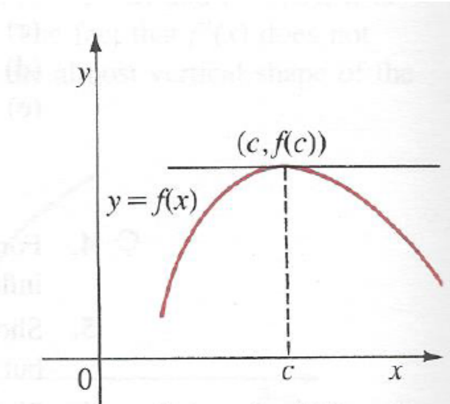
Note: $y' = 4(x-4)^2(x-1)$ and $y'' = 12(x-4)(x-2)$

Second Derivative Test

- If $f\,'(c) = 0$ and $f\,''(c) > 0$ and f is concave upward, then f has a local minimum at c.
- If $f\,'(c) = 0$ and $f\,''(c) < 0$ and f is concave downward, then f has a local maximum at c.
- If $f\,''(c) = 0$ then $x = c$ can be a local maximum, local minimum or neither (i.e., students must use the first derivative test to classify the critical point).



$f'(c) = 0, f''(c) > 0$



$f'(c) = 0, f''(c) < 0$

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Classify critical points as local maxima or local minima using

(1) first derivative or

(2) second derivative

Example: Determine the extrema for the function $f(x) = x^3 - 3x + 2$

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Example:

Sketch the graph of a continuous function that satisfies all of the following conditions:

$f(0) = f(3) = 0$

$f'(-1) = f'(1) = -2$

$f'(x) < 0 \text{ for } x < -1 \text{ and } 0 < x < 1$

$f'(x) > 0 \text{ for } -1 < x < 0 \text{ and } x > 1$

$f''(x) > 0 \text{ for } x < 3$

$f''(x) < 0 \text{ for } x > 3$

$\lim_{x \to \infty} f(x) = 1$

$\lim_{x \to -\infty} f(x) = 0$

Example:

Sketch the graph of a continuous function that satisfies all of the following conditions:

$f'(x) < 0 \; on \; (1, \infty)$

$f'(x) > 0 \; on \; (-\infty, 1)$

$f''(x) > 0 \;\; on \; (-\infty, -2) \; and \; (2, \infty)$

$f''(x) < 0 \;\; on \; (-2, 2)$

$\lim_{x \to -\infty} f(x) = -2$

$\lim_{x \to \infty} f(x) = 0$

➞

Example:

Sketch the graph of a continuous function that satisfies all of the following conditions:

$f'(0) = f'(2) = f'(4) = 0$

$f'(x) < 0$ *if* $0 < x < 2$ *or* $x > 4$

$f'(x) > 0$ *if* $x < 0$ *or* $2 < x < 4$

$f''(x) > 0$ *if* $1 < x < 3$

$f''(x) < 0$ *if* $x < 1$ *or* $x > 3$