CHAPTER 7

Calculus of Exponential and Logarithmic Functions

(i) The number e ⟶ Euler's number

(ii) Evaluate $\lim_{h\to\infty}\left(1+\frac{1}{h}\right)^{h}$

(iii) Inverse Functions $y=a^{x}$ and $y=\log_a x$

$y=e^{x}$ and $y=\ln x$

(iv) Derivatives of $y=a^{x}$ $\quad y=\log_a x$

$y=e^{x}$ $\quad y=\ln x$

(v) Logarithmic Differentiation

(vi) Proofs

⟶

Exponential and Logarithmic Functions

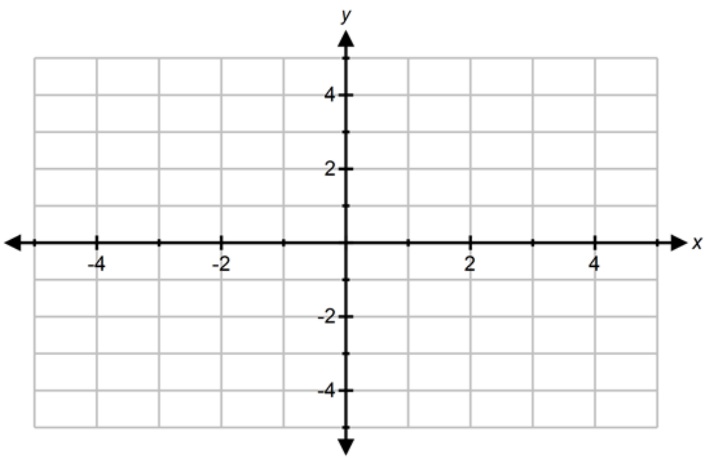
What is the inverse of the exponential function $y = 2^x$?

$y = 2^x$

X	Y

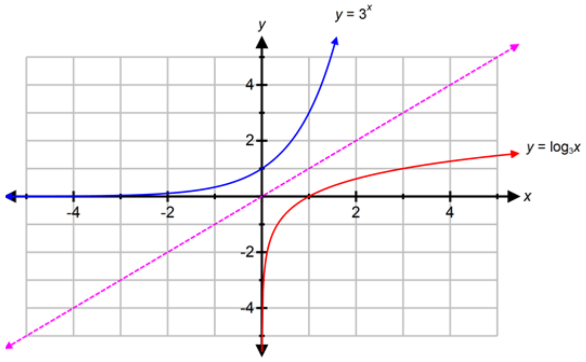
Inverse Function $\longrightarrow$

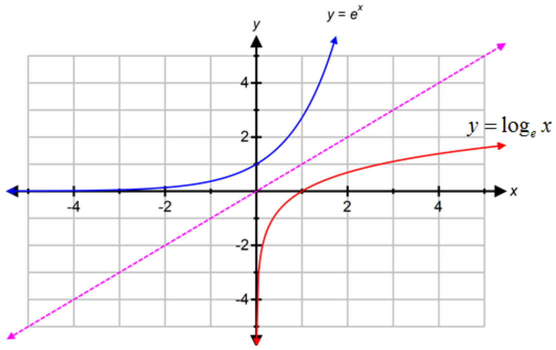
X	Y



Rule: $\log_b n = e \longleftrightarrow b^e = n$

Exponential and Logarithmic Functions





What is the value of e?

$e = 2.71828182...$

irrational number

$$e = \lim_{h \to \infty} \left(1 + \tfrac{1}{h}\right)^h$$

By letting h take on larger and larger values, the value is close to the decimal value for e.

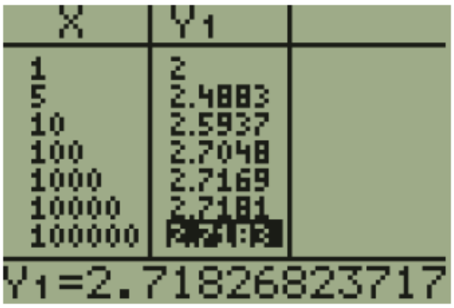
$$\left(1+\frac{1}{2}\right)^2 = 2.25$$

$$\left(1+\frac{1}{5}\right)^5 =$$

$$\left(1+\frac{1}{10}\right)^{10} =$$

$$\left(1+\frac{1}{100}\right)^{100} =$$

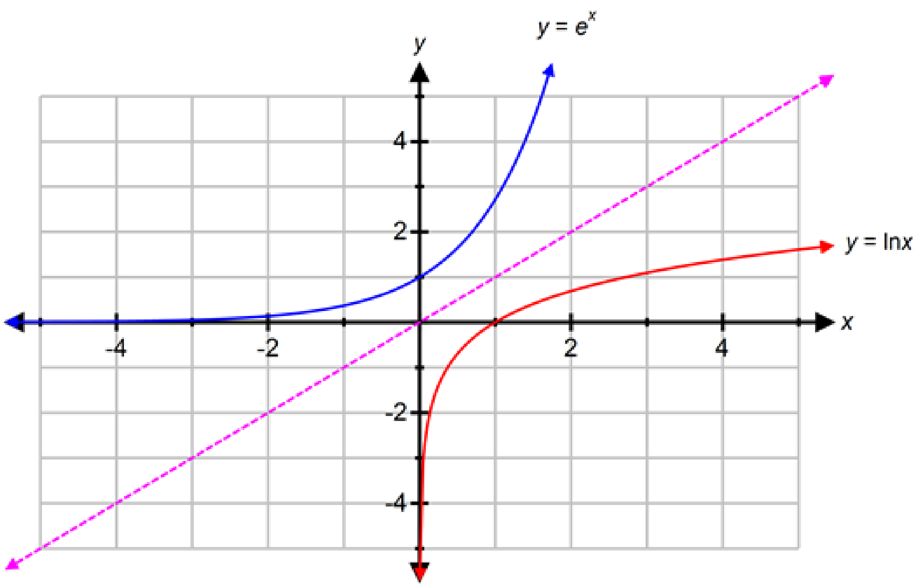
$$\left(1+\frac{1}{1000}\right)^{1000} =$$



Common Logarithms and Natural Logarithms

Common Logarithms	Natural Logarithms
↓	↓
base 10	base e
↓	↓
$y = \log_{10} x$	$y = \log_e x$
↓	↓
$y = \log x$	$y = \ln x$

Natural logarithm $y = \ln x$ is the <u>inverse</u> of the exponential function $y = e^x$



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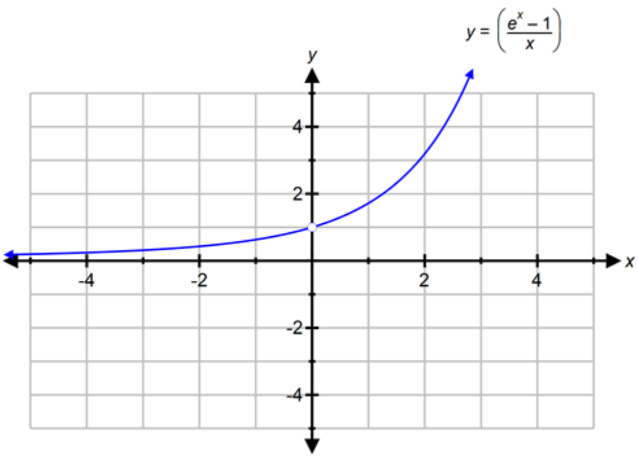
Derivatives

In order to compute the derivative of the exponential function $y = e^x$, we must first determine the value of the limit $\lim_{h \to 0} \frac{e^h - 1}{h}$ using a table of values.

h	$\frac{e^h - 1}{h}$
-0.01	0.995
-0.001	0.9995
0	?
0.001	1.0005
0.01	1.005



$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

→

Derivative of $y = e^x$

(i) Differentiate $y = e^x$

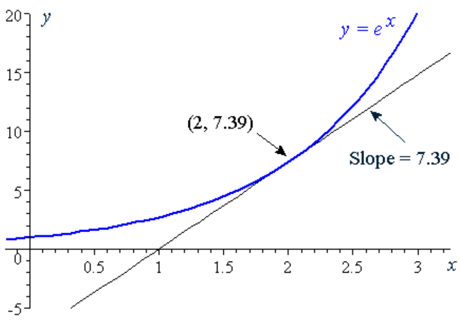
$f(x) = e^x$ Use definition of limit and the special limit $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Chain Rule: $\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx}$

Technology

http://www.flashandmath.com/mathlets/calc/derplot/derplot.html



Example 1

Determine y' for the following:

(a) $y = e^{3x-7x^2}$

(b) $y = e^{\sec(3x)}$

(c) $y = e^{x(\sin x)}$

(d) $y = \dfrac{e^{2x}}{\sin(e^x)}$

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(e) $y = \dfrac{e^{2x}}{\arctan(x)}$

(f) $y = \tan^{-1}(e^{2x})$

(g) $y = \dfrac{e^{\sqrt{x}}}{x}$

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(h) $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

(i) $y = xe^{\tan x}$

(j) $y = \cos(e^{x^3})$

(k) $y = \csc(x^2 e^x)$

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Derivative of $y = \ln x$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

(i) Differentiate $y = \ln x$

$y = \ln x$ change to exponential form and use implicit differentiation

Chain Rule: $\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}$ (where u is a function of x)

Example 2

Determine y' for the following:

(a) $y = \dfrac{\ln x}{\sin x}$

(b) $y = \ln(\cos^2 x)$

(c) $y = x^2 \ln(x)$

(d) $y = \cot(\ln(x^2 - 3))$

→

(e) $y = \ln(\arcsin\sqrt{x})$

(f) $y = \dfrac{\sin(x)\ln x}{e^{\sin x}}$

(g) $y = \dfrac{\sin(x)\ln x}{e^{\sin x}}$

(h) $y = \ln(3x-4)^2$

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Simplify the function using the Logarithmic Properties

$$\ln(xy) = \ln x + \ln y$$

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

$$\ln(x^r) = r \ln x$$

(i) $y = \ln(3x-4)^2$

(j) $y = \ln\left(\frac{x}{3x-4}\right)$

(k) $y = \ln\left(\frac{x^2-2}{2x^2-1}\right)$

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(l) $y = \ln[(x+1)^2(4x-1)^3]$

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