Sample CPT

Multiple Choice

Identify the choice that best completes the statement or answers the question.

1. Evaluate: $\lim\limits_{x \to -5} \dfrac{x^2 - 25}{x^3 + 5x^2 + 5x + 25}$

   A) 1

   B) -1

   C) $-\dfrac{1}{3}$

   D) $\dfrac{1}{5}$

   E) does not exist

2. Evaluate: $\lim\limits_{x \to 2} \dfrac{-3 + \sqrt{4x + 1}}{10 - 5x}$

   A) $-\dfrac{2}{15}$

   B) $-\dfrac{4}{5}$

   C) $-\dfrac{1}{5}$

   D) $\dfrac{2}{3}$

   E) does not exist

Name: ________________________ **ID: A**

3. Evaluate: $\lim_{x \to 3^+} \frac{|3-x|}{x^2-6x+9}$

   A) 0

   B) $-\frac{1}{9}$

   C) ∞

   D) $-\infty$

   E) does not exist

4. Evaluate: $\lim_{x \to -\infty} \frac{2x-1}{x-\sqrt{9x^2+7}}$

   A) -1

   B) $\frac{1}{2}$

   C) 0

   D) 2

   E) does not exist

5. Differentiate: $y = \csc\left(x^2 e^x\right)$

   A) $-\csc\left(x^2 e^x\right)\cot\left(x^2 e^x\right)$

   B) $-\csc\left(x^2 e^x\right)$

   C) $-xe^x(x+2)\csc\left(x^2 e^x\right)\cot\left(x^2 e^x\right)$

   D) $-xe^x(x+2)\csc\left(x^2 e^x\right)$

   E) $-2xe^x\csc\left(x^2 e^x\right)\cot\left(x^2 e^x\right)$

   F) $-2xe^x\csc\left(x^2 e^x\right)$

Name: ________________________ **ID: A**

6. Differentiate: $y=\frac{\arctan(x)}{1+x^2}$

A) $\frac{1}{2x\left(1+x^2\right)}$

B) $\frac{\sec^2(x)}{2x}$

C) $\frac{1-2x\cdot\arctan(x)}{\left(1+x^2\right)^2}$

D) $\frac{2x\cdot\arctan(x)-1}{\left(1+x^2\right)^2}$

E) $\frac{1-2x\cdot\arctan(x)}{1+x^2}$

F) $\frac{2x\cdot\arctan(x)-1}{1+x^2}$

7. Differentiate: $y=\sin\left(e^{x^4}\right)$

A) $\cos\left(e^{x^4}\right)$

B) $4x^3\cos\left(e^{x^4}\right)$

C) $\cos\left(4x^3e^{x^4}\right)$

D) $4x^3e^{x^4}\cos\left(e^{x^4}\right)$

E) $e^{x^4}\cos\left(e^{x^4}\right)$

F) $4x^3e^{x^4}\sin\left(e^{x^4}\right)\cos\left(e^{x^4}\right)$

8. Differentiate: $y = x^{\sin(x)}$
   A) $x^{\sin(x)-1}$
   B) $\cos(x)\cdot x^{\sin(x)-1}$
   C) $x^{\sin(x)}\cdot \ln\left(\sin(x)\right)$
   D) $x^{\sin(x)}\cdot \dfrac{\cos(x)}{x}$
   E) $x^{\sin(x)}\left[\cos(x)\cdot \ln(x)+1\right]$
   F) $x^{\sin(x)}\left[\cos(x)\cdot \ln(x)+\dfrac{\sin(x)}{x}\right]$

9. Differentiate: $xy^2 = 2x - y$
   A) $\dfrac{2-y^2}{2xy+1}$
   B) $\dfrac{2}{2xy+1}$
   C) $\dfrac{2-y^2}{2x+1}$
   D) $\dfrac{2}{2x+1}$
   E) $\dfrac{2}{y^2}-\dfrac{1}{xy}$
   F) $-\dfrac{4}{y^3}+\dfrac{1}{x^2y}$

Name: ________________________ **ID: A**

Completion
Complete each statement.

10. Consider the function

$$f(x) = \begin{cases} \dfrac{4x+8}{x^2-2x-8}, \text{ for } x \le 0 \\ x^2-2x, \text{ for } x > 0 \end{cases}$$

a)

Use the definition of continuity to determine whether $f(x)$ is continuous at $x = 0$. If it is not, is the discontinuity removable or non-removable?

b)

Use the definition of continuity to determine all other points at which $f(x)$ is not continuous. classify any discontinuities as removable or non-removable.

11. a)

Use the definition of a derivative to differentiate $f(x) = \dfrac{2x}{5-x}$

b)

Find the equation of the tangent line to the curve $y = \dfrac{2x}{5-x}$ at the point $x = 4$.

12. A spotlight on the ground shines on a wall 12 metres away. If a man 2 metres tall walks from the spotlight toward the building at a speed of 1 metre per second, how fast is his shadow on the building shrinking when he is 4 metres from the building?

13. A box is to be manufactured such that its width is exactly three times its length and its volume is 36 cubic metres. Assuming that the box is closed on all sides, determine the smallest amount of material needed for its construction.

14. Consider the function; $f(x) = \dfrac{9\left(x^2 - 16\right)}{(x+8)^2}$ with derivatives;

$$f'(x) = \frac{144(x+2)}{(x+8)^3} \quad \text{and} \quad f''(x) = \frac{-288(x-1)}{(x+8)^4}$$

a) Find the vertical asymptotes of the graph of $f(x)$, if any.

b) Find the horizontal asymptotes of graph of $f(x)$, if any.

c) Find the x and y-intercepts of the graph of $f(x)$, if any.

d) Determine the intervals on which $f(x)$ is increasing or dereasing and classify any relative (local) extrema.

e) Determine the intervals on which f(x) is concave upward or concave downward, and identify any points of inflection.

f) Sketch the graph of $f(x)$ and label your graph carefully.

15. Use the definition of the derivative to prove that if f(x) and g(x) are differentiable functions then $[f(x)g(x)]' = f'(x)g(x) + g'(x)f(x)$.

Sample CPT
Answer Section

MULTIPLE CHOICE

1. ANS: C

Evaluate: $\lim_{x \to -5} \frac{x^2 - 25}{x^3 + 5x^2 + 5x + 25}$

Solution:

$$\lim_{x \to -5} \frac{(x-5)(x+5)}{\left(x^3 + 5x^2\right) + (5x + 25)} = \lim_{x \to -5} \frac{(x-5)(x+5)}{x^2(x+5) + 5(x+5)}$$

$$= \lim_{x \to -5} \frac{(x-5)(x+5)}{(x+5)\left(x^2+5\right)} = \lim_{x \to -5} \frac{(x-5)}{\left(x^2+5\right)} = \frac{(-5-5)}{\left((-5)^2+5\right)} = \frac{-10}{30} = -\frac{1}{3}$$

PTS: 1

2. ANS: A

Evaluate: $\lim_{x \to 2} \frac{-3 + \sqrt{4x+1}}{10 - 5x}$

Solution:

$$\lim_{x \to 2} \frac{-3 + \sqrt{4x+1}}{10 - 5x} \cdot \frac{-3 - \sqrt{4x+1}}{-3 - \sqrt{4x+1}} = \lim_{x \to 2} \frac{9 - (4x+1)}{(10-5x)\left(-3 - \sqrt{4x+1}\right)}$$

$$= \lim_{x \to 2} \frac{-4x + 8}{(10-5x)\left(-3 - \sqrt{4x+1}\right)} = \lim_{x \to 2} \frac{-4(x-2)}{-5(x-2)\left(-3 - \sqrt{4x+1}\right)}$$

$$= \lim_{x \to 2} \frac{-4}{-5\left(-3 - \sqrt{4x+1}\right)} = \frac{-4}{-5\left(-3 - \sqrt{4(2)+1}\right)} = \frac{-4}{-5(-6)} = -\frac{2}{15}$$

PTS: 1

3. ANS: C

Evaluate: $\lim\limits_{x \to 3^+} \dfrac{|3-x|}{x^2-6x+9}$

Solution:

Note: $|3-x| = |-1(x-3)| = |x-3|$ and $|x-3| = \begin{cases} x-3, & x \geq 3 \\ -(x-3), & x < 3 \end{cases}$

So in this case the limit is conisdering values greater than 3 so we can rewrite the limit as;

$$\lim_{x \to 3^+} \frac{x-3}{x^2-6x+9} = \lim_{x \to 3^+} \frac{x-3}{(x=3)(x-3)} = \lim_{x \to 3^+} \frac{1}{(x-3)} = \frac{1}{0^+} = \infty$$

PTS: 1

4. ANS: B

Evaluate: $\lim\limits_{x \to -\infty} \dfrac{2x-1}{x-\sqrt{9x^2+7}}$

Solution:

$$\lim_{x \to -\infty} \frac{2x-1}{x-\sqrt{9x^2+7}} = \frac{-\infty}{-\infty}$$ Indeterminant Form.

$$\lim_{x \to -\infty} \frac{2x-1}{x-\sqrt{9x^2+7}} = \lim_{x \to -\infty} \frac{2x-1}{x-(-x)\sqrt{9+\frac{7}{x^2}}} = \lim_{x \to -\infty} \frac{x\left(2-\frac{1}{x}\right)}{x\left(1+\sqrt{9+\frac{7}{x^2}}\right)}$$

$$= \lim_{x \to -\infty} \frac{\left(2-\frac{1}{x}\right)}{\left(1+\sqrt{9+\frac{7}{x^2}}\right)} = \frac{(2-0)}{\left(1+\sqrt{9+0}\right)} = \frac{2}{4} = \frac{1}{2}$$

PTS: 1

5. ANS: C

Differentiate: $y = \csc\left(x^2 e^x\right)$

Solution:

$y' = -\csc\left(x^2 e^x\right)\cot\left(x^2 e^x\right) \cdot \left[x^2 e^x\right]'$

$= -\csc\left(x^2 e^x\right)\cot\left(x^2 e^x\right) \cdot \left[2xe^x + x^2 e^x\right]$

$= -xe^x\,(x+2)\csc\left(x^2 e^x\right)\cot\left(x^2 e^x\right)$

PTS: 1

6. ANS: C

Differentiate: $y = \dfrac{\arctan(x)}{1+x^2}$

Solution:

$$y' = \frac{\left(\dfrac{1}{1+x^2}\right)\cdot\left(1+x^2\right) - \arctan(x)\cdot(2x)}{\left(1+x^2\right)^2} = \frac{1-2x\cdot\arctan(x)}{\left(1+x^2\right)^2}$$

PTS: 1

7. ANS: D

Differentiate: $y = \sin\left(e^{x^4}\right)$

Solution:

$$y' = \cos\left(e^{x^4}\right)\cdot e^{x^4}\cdot 4x^3 = 4x^3 e^{x^4}\cos\left(e^{x^4}\right)$$

PTS: 1

8. ANS: F

Differentiate: $y = x^{\sin(x)}$

Solution:

$y = x^{\sin(x)}$

$\ln y = \ln x^{\sin(x)}$

$\ln y = \sin(x) \cdot \ln x$

Differentiate Implicitly;

$\frac{1}{y} \cdot y' = \cos(x) \cdot \ln x + \sin(x) \cdot \frac{1}{x}$

$y' = x^{\sin(x)} \left[\cos(x) \cdot \ln x + \frac{\sin(x)}{x} \right]$

PTS: 1

9. ANS: A

Differentiate: $xy^2 = 2x - y$

Solution:

$\mathrm{y}^2 + x \cdot 2y \cdot y' = 2 - y'$

$x \cdot 2y \cdot y' + y' = 2 - \mathrm{y}^2$

$y'\left(2xy + 1\right) = 2 - \mathrm{y}^2$

$y' = \frac{2 - \mathrm{y}^2}{2xy + 1}$

PTS: 1

COMPLETION

10. ANS:

Solution:

a) $f(x)$ is continuous at $x = 0$ if;

$\lim_{x \to 0} f(x)$ exists and is equal to $f(0)$.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{4x+8}{x^2-2x-8} = \frac{0+8}{0-0-8} = -1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} x^2 - 2x = 0 - 0 = 0$$

Since the limit from the left does not equal the limit from the right the overall limit as x approaches zero, does not exist so the function is not continuous at $x = 0$. With both the right-hand and left hand limits being different we have non-removable discontinuity (or jump discontinuity).

b) Other disconinuities;

Since we have peicewise function which is comprised of a rational function and polynomial function it will be continuous everywhere except for the non-permissible values for the rational function and the point where the two functions meet. We have already tested this value, at $x = 0$, and found it to be discontinuous. We will now need to consider the non-permissible values for the rational function piece;

$\dfrac{4x+8}{x^2-2x-8}$ is undefined when its denominator is zero;

$x^2 - 2x - 8 = 0 \quad \Rightarrow \quad (x-4)(x+2) = 0$ so we have discontinuities at;

$x = 4$ and $x = -2$. Since we only consider the rational function for $x \leq 0$, we only need to test the status of $x = -2$;

$$\lim_{x \to -2} \frac{4x+8}{x^2-2x-8} = \lim_{x \to -2} \frac{4(x+2)}{(x-4)(x+2)} = \lim_{x \to -2} \frac{4}{(x-4)} = \frac{4}{-2-4} = -\frac{2}{3}$$

Since, the limit exist but the function is undefined at $x = -2$, we have a removable discontinuity at $x = -2$.

So, overall *f*(*x*) is discontinuous at two x-values;

$x = -2$ (removable, hole)

$x = 0$ (non-removable or jump discontinuity)

PTS: 1

11. ANS:

a)

Use the definition of a derivative to differentiate $f(x)=\dfrac{2x}{5-x}$

Solution:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\dfrac{2(x+h)}{5-(x+h)}-\dfrac{2x}{5-x}}{h}$$

$$=\lim_{h\to 0}\frac{2(x+h)(5-x)-2x(5-x-h)}{h(5-x-h)(5-x)}$$

$$=\lim_{h\to 0}\frac{10x-2x^2+10h-2hx-10x+2x^2+2xh}{h(5-x-h)(5-x)}$$

$$=\lim_{h\to 0}\frac{10h}{h(5-x-h)(5-x)}$$

$$=\lim_{h\to 0}\frac{10}{(5-x-h)(5-x)}=\frac{10}{(5-x-0)(5-x)}=\frac{10}{(5-x)^2}$$

b)

Find the equation of the tangent line to the curve $y=\dfrac{2x}{5-x}$ at the point $x=4$.

Solution:

Since $f'(x)=\dfrac{10}{(5-x)^2}$, $\quad f'(4)=\dfrac{10}{(5-4)^2}=10$

$$f(4)=\frac{2(4)}{5-4}=\frac{8}{1}=8$$

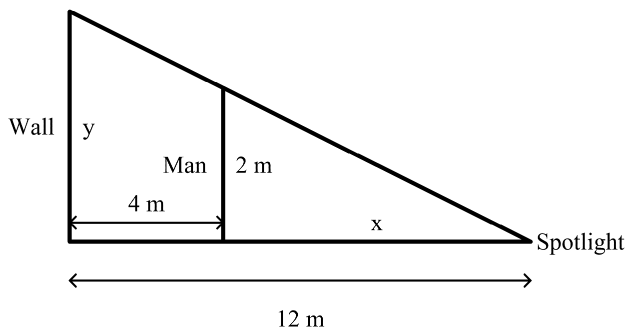
The equation of the tangent line would be;

$$y-y_1=m\left(x-x_1\right)$$

$y-8=10(x-4)$ or $y=10x-32$

PTS: 1

12. ANS:
Solution:



At any time when the man is walking towards the wall the following is true;

$$\frac{12}{y} = \frac{x}{2} \quad \text{or} \quad 24 = xy$$

We know $\frac{dx}{dt} = 1$ we are looking for $\frac{dy}{dt}$.

Lets differentiate the above expression implicitly with respect to t;

$$24 = xy$$

$$0 = x'y + xy' \quad \text{or} \quad y' = -\frac{x'y}{x}$$

$$\frac{dy}{dt} = -\frac{dx}{dt} \cdot \frac{y}{x}$$

At the instance when the man is 4 m from the wall, $x = 8$ m and

$24 = 8y$ or $y = 3$ m

so,

$$\frac{dy}{dt} = -\frac{dx}{dt} \cdot \frac{y}{x} = -(1)\left(\frac{3}{8}\right) = -\frac{3}{8} \text{ m/s}$$

So the man's shadow is decreasing at a rate of $\frac{3}{8}$ m/s.

PTS: 1

13. ANS:

Solution:

$V = l \times w \times h$ and $w = 3l$

$V = l \times 3l \times h = 3l^2 h$

$V = 36 = 3l^2 h \quad \Rightarrow \quad h = \dfrac{12}{l^2}$

We need to minimize the surface area;

$SA = 2(lw + lh + wh)$

$SA = 2(l \cdot 3l + lh + 3l \cdot h) = 2\left(3l^2 + 4l \cdot \dfrac{12}{l^2}\right) = 2\left(3l^2 + \dfrac{48}{l}\right) = 6l^2 + \dfrac{96}{l}$

$SA' = 12l - \dfrac{96}{l^2}$

SA could be a minimum when $SA' = 0$;

$12l - \dfrac{96}{l^2} = 0 \quad \Rightarrow \quad 12l^3 - 96 = 0 \quad \Rightarrow \quad l^3 = 8 \quad \Rightarrow \quad l = 2 \text{ m}$

Is this value of $l = 2$ m an actual minimum;

$SA'' = 12 + \dfrac{192}{l^3}$ at $l = 2$ m the second derivative is $12 + \dfrac{192}{2^3} = 12 + 24 = 36 > 0$

(Concave up) making it a local minimum and with no other points to consider, it is an absolute minimum.

So the smallest amount of material needed is when the length is 2 m, height is 3 m and width is 6 m giving us a total surface area of;

$2(3 \times 2 + 2 \times 6 + 3 \times 6) = 72 \text{ m}^2$

PTS: 1

14. ANS:

Consider the function; $f(x)=\dfrac{9\left(x^2-16\right)}{(x+8)^2}$ with derivatives;

$f'(x)=\dfrac{144(x+2)}{(x+8)^3}$ and $f''(x)=\dfrac{-288(x-1)}{(x+8)^4}$

a) Find the vertical asymptotes of the graph of $f(x)$, if any.

Lets first consider the non-permissible values;

$(x+8)^2=0\;\; x=-8$ is a non-permissible value and since it makes the denominator zero only it must be a vertical asymptote. We will validate using limits;

$$\lim_{x\to -8^+}\frac{9\left(x^2-16\right)}{(x+8)^2}=\frac{432}{0^+}=+\infty$$

$$\lim_{x\to -8^-}\frac{9\left(x^2-16\right)}{(x+8)^2}=\frac{432}{0^+}=+\infty$$

Therefore, at, $x=-8$ there is a vertical asymptote.

b) Find the horizontal asymptotes of graph of $f(x)$, if any.

$$\lim_{x\to -\infty}\frac{9\left(x^2-16\right)}{(x+8)^2}=\lim_{x\to -\infty}\frac{9x^2-144}{x^2+16x+64}=\lim_{x\to -\infty}\frac{9-\frac{144}{x^2}}{1+\frac{16}{x}+\frac{64}{x^2}}=\frac{9}{1}=9$$

$$\lim_{x\to \infty}\frac{9\left(x^2-16\right)}{(x+8)^2}=\lim_{x\to \infty}\frac{9x^2-144}{x^2+16x+64}=\lim_{x\to \infty}\frac{9-\frac{144}{x^2}}{1+\frac{16}{x}+\frac{64}{x^2}}=\frac{9}{1}=9$$

So, we have one horizontal asymptote at $y=9$

c) Find the x and y-intercepts of the graph of $f(x)$, if any.

$$f(x)=\frac{9\left(x^2-16\right)}{(x+8)^2}$$

x-intercepts;

$\left(x^2-16\right)=0,\ \ \text{x}=4$ and $x=-4$

y-intercept;

$$y=\frac{9\left(0^2-16\right)}{(0+8)^2}=-\frac{144}{64}=-\frac{9}{4}=-2.25$$

d) Determine the intervals on which $f(x)$ is increasing or dereasing and classify any relative (local) extrema.

$$f'(x)=\frac{144(x+2)}{(x+8)^3}$$

Critical numbers are; $x=-2$ and $x=-8$

Test;

$x=-9,\ \ f'(-9)=\dfrac{144(-9+2)}{(-9+8)^3}>0$ So, $f(x)$ is increasing $\text{x}\in\left(-\infty,-8\right)$

$x=-3,\ \ f'(-3)=\dfrac{144(-3+2)}{(-3+8)^3}<0$ So, $f(x)$ is decreasing $\text{x}\in\left(-8,-2\right)$

$x=0,\ \ f'(0)=\dfrac{144(0+2)}{(0+8)^3}>0$ So, $f(x)$ is increasing $\text{x}\in\left(-2,\infty\right)$

Therefore, $x=-2$ is a local minimum (decreasing on the left and increasing on the right). Since the only vertical asymptote at $x=-8$ is approaching $+\infty$ on both sides this extrema at $x=-2$ is an absolute minimum.

When $x=-2,\ \ y=\dfrac{9\left((-2)^2-16\right)}{(-2+8)^2}=-\dfrac{108}{36}=-3$

e) Determine the intervals on which f(x) is concave upward or concave downward, and identify any points of inflection.

$$f''(x) = \frac{-288(x-1)}{(x+8)^4}$$

Hypercritical points;

$x = 1$ and $x = -8$

Test;

$x = -9,\ f''(-9) = \frac{-288(-9-1)}{(-9+8)^4} > 0$ So, f(x) is concave up when $x \in (-\infty, -8)$

$x = 0,\ f''(0) = \frac{-288(0-1)}{(0+8)^4} > 0$ So, f(x) is concave up when $x \in (-8, 1)$

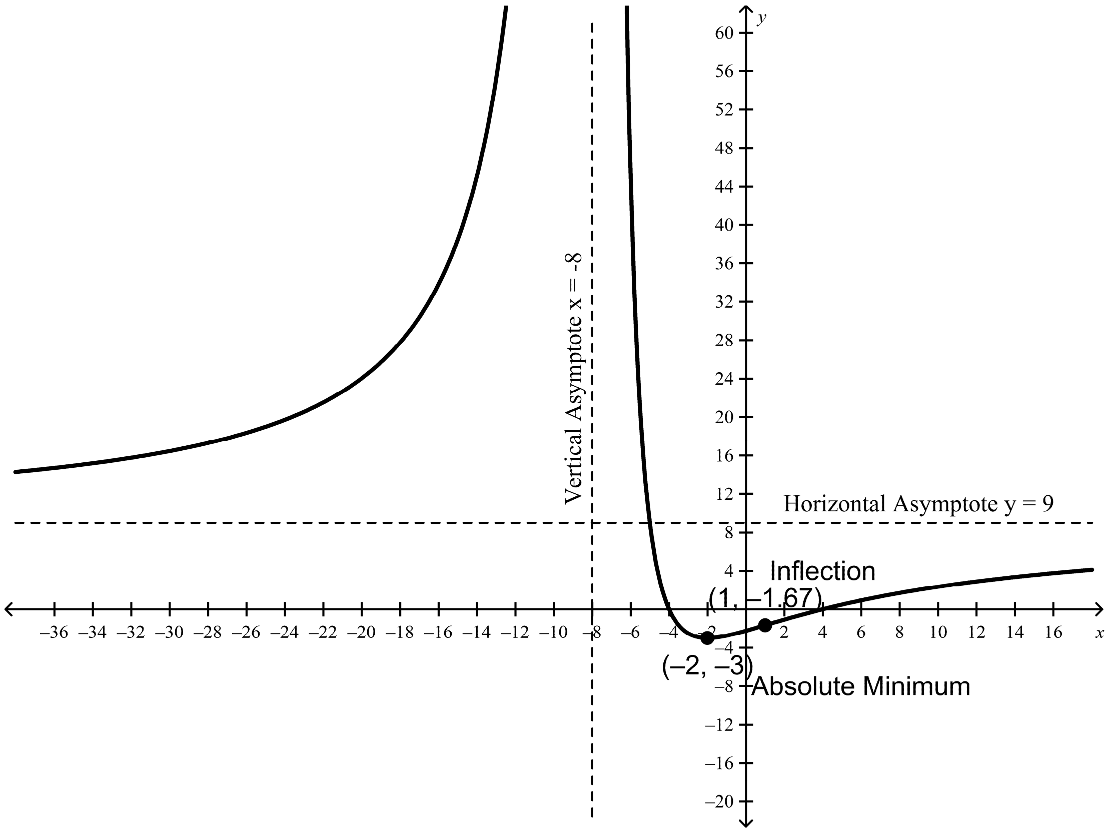
$x = 2,\ f''(20) = \frac{-288(2-1)}{(2+8)^4} < 0$ So, f(x) is concave down when $x \in (1, \infty)$

Since concavity changes at $x = 1$, then $x = 1$ is a point of inflection.

When $x = 1,\ y = \frac{9\left(1^2 - 16\right)}{(1+8)^2} = -\frac{135}{81} = -\frac{5}{3}$

So, a point of inflection at $\left(1, -\frac{5}{3}\right)$

f) Sketch the graph of *f*(*x*) on the axes provided. Label your graph carefully.



PTS: 1

15. ANS: $[f(x)g(x)]' = \lim_{h \to 0} \dfrac{f(x+h)g(x+h) - f(x)g(x)}{h}$

The trick now is to introduce a zero term; $-f(x+h)g(x) + f(x+h)g(x)$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)\left[g(x+h) - g(x)\right]}{h} + \lim_{h \to 0} \frac{g(x)\left[f(x+h) - f(x)\right]}{h}$$

$$= \lim_{h \to 0} f(x+h) \cdot \lim_{h \to 0} \frac{\left[g(x+h) - g(x)\right]}{h} + \lim_{h \to 0} g(x) \cdot \lim_{h \to 0} \frac{\left[f(x+h) - f(x)\right]}{h}$$

$$= f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

PTS: 1